Int J Theor Phys (2008) 47: 520–525
DOI 10.1007/s10773-007-9476-1

Generalized Uncertainty Principle and Correction Value to the Kerr Black Hole Entropy

Zhang Ya · Hu Shuang-Qi · Zhao Ren · Li Huai-Fan

Received: 10 April 2007 / Accepted: 16 June 2007 / Published online: 6 July 2007

Abstract Recently, there has been much attention devoted to resolving the quantum corrections to the Bekenstein-Hawking black hole entropy. In particular, many researchers have expressed a vested interest in the coefficient of the logarithmic term of the black hole entropy correction term. In this paper, we calculate the correction value of the black hole entropy by utilizing the generalized uncertainty principle and obtain the correction term caused by the generalized uncertainty principle. Because in our calculation we think that the Bekenstein-Hawking area theorem is still valid after considering the generalized uncertainty principle, we derive that the coefficient of the logarithmic term of the black hole entropy correction term is positive. This result is different from the known result at present. Our method is valid not only for single horizon spacetime but also for spin axial symmetric spacetimes with double horizons. In the whole process, the physics idea is clear and calculation is simple. It offers a new way for studying the entropy correction of the complicated spacetime.

Keywords Generalized uncertainty principle · Black hole entropy · Area theorem

1 Introduction

One of the most remarkable achievements in gravitational physics was the realization that black holes have temperature and entropy [1–4]. There is a growing interest in the black hole entropy. Since entropy has statistical physics meaning in the thermodynamic system and it is related to the number of microstates of the system. However, in Einstein general

Z. Ya
Mechanical and Electronic Engineering Institute, North University of China, Taiyuan 030051, China

H. Shuang-Qi · Z. Ren (✉)
Chemical Industry and Ecology Institute, North University of China, Taiyuan 030051, China
e-mail: zhaoren2969@yahoo.com.cn

Z. Ren · L. Huai-Fan
Department of Physics, Shanxi Datong University, Datong 037009, China

relativity theory, the black hole entropy is a pure geometry quantity. If we compare the black hole with the thermodynamic system, we will find an important difference. A black hole is a vacancy with strong gravitation. But the thermodynamic system is composed of atoms and molecules. Based on the microstructure of thermodynamic systems, we can explain thermodynamic property by statistic mechanics of its microcosmic elements. Whether the black hole has interior freedom degree corresponding the black hole entropy [5]? Let us suppose that the Bekenstein-Hawking entropy has a definite statistical meaning. Then how could we identify these microstates and, even more optimistically, counting them [6–8]? This is a key problem in studying the black hole entropy.

Recently, both string theory and loop quantum gravity have been had successful explaining the entropy-area "law" [6–8] statistically. However, who might actually prefer if there was only one fundamental theory? It is expected to choose it by quantum correction term of the black hole entropy. Therefore, studying the black hole entropy correction value becomes the focus of attention. Many ways of discussing the black hole entropy correction value have emerged [6–15]. But the exact value of coefficient of the logarithmic term in the black hole entropy correction term is not known.

Since we discuss the black hole entropy, we need study the quantum effect of the black hole. When we discuss radiation particles or absorption ones, we should consider the uncertainty principle. However, as gravity is turned on, the "conventional" Heisenberg relation is no longer fully satisfied. The generalized uncertainty principle will replace it. In this paper, we discuss the black hole entropy correction value by the generalized uncertainty principle. There is no restriction to spacetimes in the method given by us. So our result has general meaning. We will discuss three kinds of representative spacetimes. The paper is organized as follows. Section 2 describes Schwarzschild spacetime. Section 3 discusses spin axial symmetric Kerr spacetime with double horizons. We take the simple function form of temperature ($c = \hbar = G = K_B = 1$).

2 Schwarzschild Black Hole

The linear element of Schwarzschild black hole spacetime:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{1}$$

Hawking radiation temperature T, horizon area A and entropy S are respectively

$$T = \frac{1}{4\pi r_H} = \frac{1}{8\pi M}, \qquad A = 4\pi r_H^2 = 16\pi M^2, \qquad S = \pi r_H^2 = 4\pi M^2, \tag{2}$$

where $r_H = 2M$ is the location of the black hole horizon.

Now for a black hole absorbing (radiating) a particle of energy $\Delta M \approx c\Delta p$, the increase (decrease) in the horizon area can be expressed as

$$dA = 8\pi r_H dr_H = 32\pi M dM. \tag{3}$$

Because the discussed black hole radiation is a quantum effect, the particle of energy dM should satisfy Heisenberg uncertainty relation.

$$\Delta x_i \Delta p_j \geq \delta_{ij}. \tag{4}$$

In gravity field Heisenberg uncertainty relation should be replaced by the generalized uncertainty principle [16–20]:

$$\Delta x_i \geq \frac{\hbar}{\Delta p_i} + \alpha^2 l_{pl}^2 \frac{\Delta p_i}{\hbar}, \tag{5}$$

where $l_{pl} = (\frac{\hbar G_d}{c^3})^{1/2}$ is the Planck length, α is a constant. From (5), we obtain

$$\frac{\Delta x_i}{2\alpha^2 l_{pl}^2}\left[1 - \sqrt{1 - \frac{4\alpha^2 l_{pl}^2}{\Delta x_i^2}}\right] \leq \frac{\Delta p_i}{\hbar} \leq \frac{\Delta x_i}{2\alpha^2 l_{pl}^2}\left[1 + \sqrt{1 - \frac{4\alpha^2 l_{pl}^2}{\Delta x_i^2}}\right]. \tag{6}$$

At $\alpha = 0$, we express (6) by Taylor series and derive

$$\Delta p_i \geq \frac{1}{\Delta x_i}\left[1 + \left(\frac{\alpha^2 l_{pl}^2}{(\Delta x_i)^2}\right) + 2\left(\frac{\alpha^2 l_{pl}^2}{(\Delta x_i)^2}\right)^2 + \cdots\right]. \tag{7}$$

From (3) and (4), the change of the area of the black hole horizon can be written as follows:

$$dA = 8\pi r_H dr_H = 32\pi M dp = 32\pi M \frac{1}{\Delta x}. \tag{8}$$

According to the generalized uncertainty principle (7) and (3), the change of the area of the black hole horizon can be rewritten as follows:

$$\begin{aligned} dA_G &= 8\pi r_H dr_H = 32\pi M dp \\ &= 32\pi M \frac{1}{\Delta x}\left[1 + \left(\frac{\alpha^2 l_{pl}^2}{(\Delta x)^2}\right) + 2\left(\frac{\alpha^2 l_{pl}^2}{(\Delta x)^2}\right)^2 + \cdots\right]. \end{aligned} \tag{9}$$

From (8) and (9), we have

$$dA_G = \left[1 + \left(\frac{\alpha^2 l_{pl}^2}{(\Delta x)^2}\right) + 2\left(\frac{\alpha^2 l_{pl}^2}{(\Delta x)^2}\right)^2 + \cdots\right] dA. \tag{10}$$

According the view of Refs. [6, 11], we take

$$\Delta x = 2r_H = 2\sqrt{\frac{A}{4\pi}}. \tag{11}$$

Substituting (11) into (10) and integrating, we derive

$$A_G = A + \alpha^2 l_{pl}^2 \pi \ln A - 2(\alpha^2 l_{pl}^2 \pi)^2 \frac{1}{A} - \cdots. \tag{12}$$

Based on Bekenstein-Hawking area law, we take $S = A/4$. Therefore, we can derive the expression of entropy after considering the generalized uncertainty principle. That is, the correction to entropy is given by

$$S_G = S + \frac{\alpha^2 l_{pl}^2 \pi}{4} \ln S + \frac{\alpha^2 l_{pl}^2 \pi}{4} \ln 4 - (\alpha^2 l_{pl}^2 \pi)^2 \frac{1}{8S} + \cdots + C, \tag{13}$$

where S is Bekenstein-Hawking entropy, C is an arbitrary constant. In the calculation, we can plus or minus an arbitrary constant C. From (13), we can calculate an arbitrary term of correction to entropy and obtain that the coefficient of the logarithmic correction term is positive. This result is different from that of Ref. [6].

3 Kerr Black Hole

The linear element in Kerr space-time is given by:

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left[(r^2 + a^2)\sin^2\theta + \frac{2Mra^2\sin^4\theta}{\rho^2}\right]d\varphi^2 - \frac{4Mra\sin^2\theta}{\rho^2}dtd\varphi, \tag{14}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, and $\Delta = r^2 - 2Mr + a^2$. The radiation temperature of the black hole is as follows

$$T_+ = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)}, \tag{15}$$

where $r_\pm = M \pm \sqrt{M^2 - a^2}$ are the locations of outer and inner horizons respectively:

$$A = 4\pi(r_+^2 + a^2). \tag{16}$$

Under the condition that a is a constant, we obtain

$$dA = 16\pi\frac{r_+^2}{r_+ - r_-}dM. \tag{17}$$

Considering the generalized uncertainty principle, we have

$$dA_G = \left[1 + \left(\frac{\alpha^2 l_{pl}^2}{(\Delta x)^2}\right) + 2\left(\frac{\alpha^2 l_{pl}^2}{(\Delta x)^2}\right)^2 + \cdots\right]dA. \tag{18}$$

Let $\Delta x = 2r_+$ (note: letting $\Delta x = 2(r_+ - r_-)$ makes no difference to the correction term of entropy), (18) can be rewritten as

$$dA_G = \left[1 + \frac{\alpha^2 l_{pl}^2\pi}{A} + \frac{\alpha^2 l_{pl}^2\pi a^2}{A^2} + 2\frac{(\alpha^2 l_{pl}^2)^2\pi^2}{A^2} + 2\frac{2(\alpha^2 l_{pl}^2)^2 a^2\pi^2}{A^3} + 2\frac{(\alpha^2 l_{pl}^2)^2 a^4\pi^2}{A^4}\cdots\right]dA. \tag{19}$$

Thus

$$A_G = A + \alpha^2 l_{pl}^2\pi\ln A - \frac{\alpha^2 l_{pl}^2\pi a^2}{A} - 2\frac{(\alpha^2 l_{pl}^2\pi)^2}{A} - \frac{2(\alpha^2 l_{pl}^2\pi a)^2}{A^2} - 2\frac{(\alpha^2 l_{pl}^2\pi a^2)^2}{3A^3} + \cdots. \tag{20}$$

In the above calculation, we have used the expression $a \ll r_+$. Based on Bekenstein-Hawking area law, we take $S = A/4$. Therefore, we can derive the expression of entropy after considering the generalized uncertainty principle. That is, the correction to entropy is given by

$$S_G = S + \frac{\alpha^2 l_{pl}^2 \pi}{4} \ln S + \frac{\alpha^2 l_{pl}^2 \pi}{4} \ln 4 - (\alpha^2 l_{pl}^2 \pi) \frac{a^2}{16S} - \frac{(\alpha^2 l_{pl}^2 \pi)^2}{8S} - \frac{(\alpha^2 l_{pl}^2 \pi a)^2}{32S^2} - \frac{(\alpha^2 l_{pl}^2 \pi a^2)^2}{384S^3} + \cdots + C, \tag{21}$$

where C is an arbitrary constant.

4 Conclusion

In summary, we have utilized the generalized uncertainty principle to demonstrate an explicit form for the correction term of the black hole entropy. From (13) and (21), for different spacetimes, coefficients of the logarithmic terms in the black hole entropy correction terms are same and are positive. This result is different from that of Ref. [6]. Although this paper and Ref. [6] both discuss the corrections caused by the generalized uncertainty principle to the black hole entropy. The difference between this paper and Ref. [6] is as follows: In (13) given by Ref. [6], the ratio between the black hole entropy and the horizon area is not quarter and there is a correction value; in our result the ratio between the black hole entropy and the horizon area is always quarter without reference to whether considering the generalized uncertainty principle or not. That is, Bekenstein-Hawking area law after considering the generalized uncertainty principle is always valid. Thus, we derive that coefficient of the logarithmic term in the black hole entropy correction term is positive. This is different with the logarithmic correction being negative.

Based on the above analysis, we calculate the correction term of the black hole entropy under the condition that Bekenstein-Hawking area law after considering the generalized uncertainty principle is valid and there are no others assumptions. So our calculation is reliable. Our method is valid not only for single horizon spacetime but also for unextreme spacetimes with outer and inner horizons. It offers a new way for studying the entropy correction of the complicated spacetime.

If we can obtain the exact value of the coefficient of the logarithmic term in the black hole entropy correction term by other method, we can not only determine the uncertainty number α in the generalized uncertainty principle but also obtain the condition that Bekenstein-Hawking area law is valid. Therefore, our results provide a new subject for studying the condition that Bekenstein-Hawking area law is valid.

Acknowledgements This project was supported by the National Natural Science Foundation of China under Grant No. 10374075 and the Shanxi Natural Science Foundation of China under Grant No. 2006011012.

References

1. Bekenstein, J.D.: Phys. Rev. D **7**, 2333 (1973)
2. Bekenstein, J.D.: Phys. Rev. D **9**, 3292 (1974)
3. Hawking, S.W.: Nature **248**, 30 (1974)
4. Hawking, S.W.: Commun. Math. Phys. **43**, 199 (1975)
5. Wang, Y.J.: In: Black Hole Physics, p. 263. Hunan Normal University Press, Changsha (2000) (in Chinese)
6. Medved, A.J.M., Vagenas, E.C.: Phys. Rev. D **70**, 124021 (2004)
7. Setare, M.R., Vagenas, E.C.: Phys. Lett. B **584**, 127 (2004)
8. Setare, M.R.: Eur. Phys. J. C **33**, 555 (2004)
9. Chatterjee, A., Majumdar, P.: Phys. Rev. Lett. **92**, 141301 (2004)

10. Kaul, R.K., Majumdar, P.: Phys. Rev. Lett. **84**, 5255 (2000)
11. Camellia, C.G.A., Arzano, M., Procaccini, A.: Phys. Rev. D **70**, 107501 (2004)
12. Chatterjee, A., Majumdar, P.: Phys. Rev. D **71**, 024003 (2005)
13. Myung, Y.S.: Phys. Lett. B **579**, 205 (2004)
14. Akbar, M.M., Das, S.: Class. Quantum Gravity **21**, 1383 (2004)
15. Das, S.: Class. Quantum Gravity **19**, 2355 (2002)
16. Chang, L.N., Minic, D., Okaruma, N., Takeuchi, T.: Phys. Rev. D **65**, 125028 (2002)
17. Hu, S.Q., Zhao, R.: Chin. Phys. **14**, 1477 (2005)
18. Li, X.: Phys. Lett. B **540**, 9 (2002)
19. Setare, M.R.: Int. J. Mod. Phys. A **21**, 1325 (2006)
20. Setare, M.R.: Phys. Rev. D **70**, 087501 (2004)